Counting and Combinatorics in Aperiodic Order

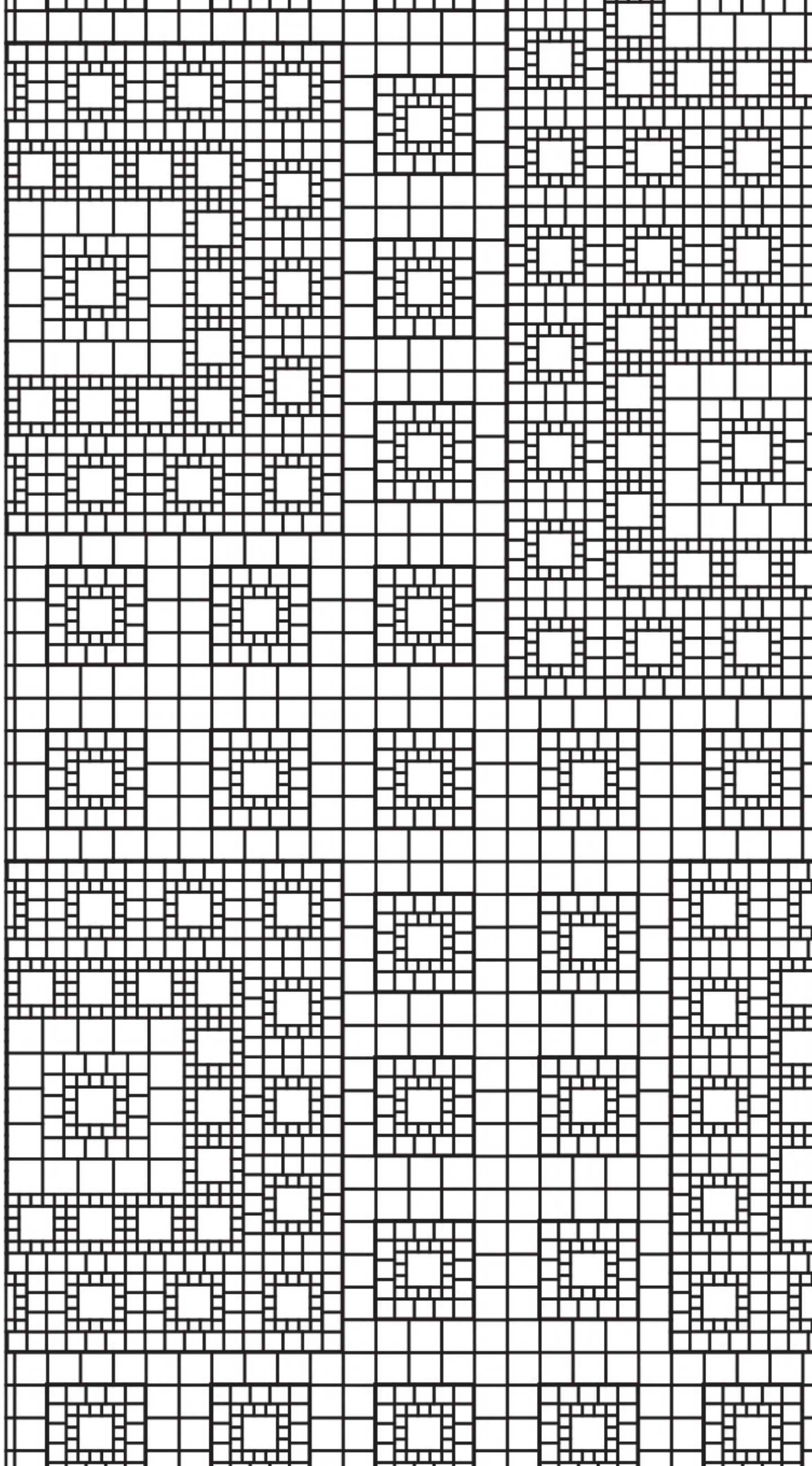
Yotam Smilansky, Rutgers

HUJI Combinatorics Seminar, 2021

Partially based on joint work with Yaar Solomon

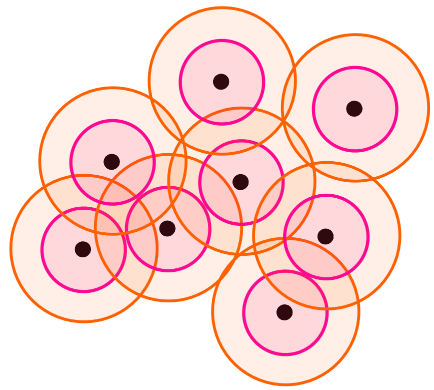
Plan of Talk

- Bounded displacement equivalence of point sets
- Substitution tilings
- Multiscale substitution tilings

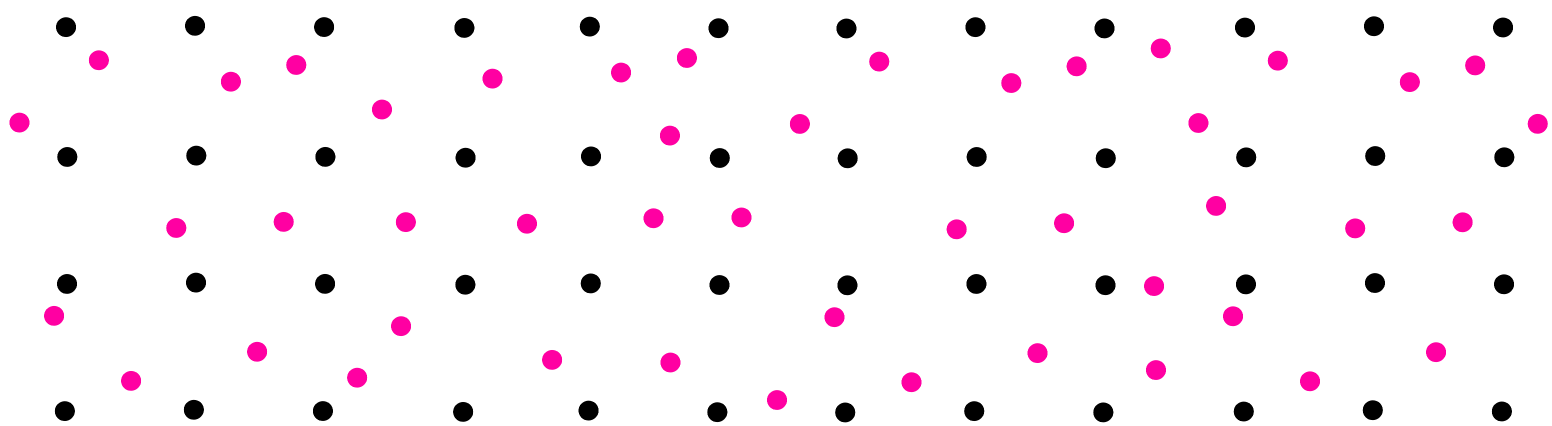


Bounded Displacement Equivalence

A uniformly discrete and relatively dense set $\Lambda \subseteq \mathbb{R}^d$ is called Delone.

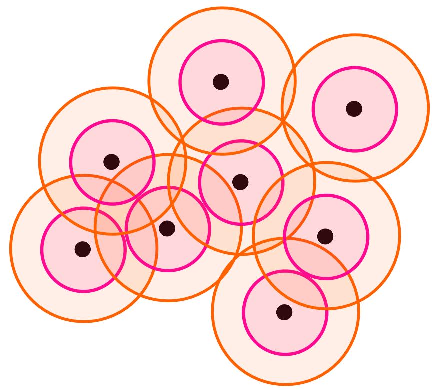


Delone sets $\Lambda, \Gamma \subseteq \mathbb{R}^d$ are bounded displacement (BD) equivalent if $\exists$ bijection $\varphi : \Lambda \to \Gamma$ that moves every point a bounded distance.



Bounded Displacement Equivalence

A uniformly discrete and relatively dense set $\Lambda \subseteq \mathbb{R}^d$ is called Delone.



Delone sets $\Lambda, \Gamma \subseteq \mathbb{R}^d$ are bounded displacement (BD) equivalent if $\exists$ bijection $\varphi : \Lambda \to \Gamma$ that moves every point a bounded distance.

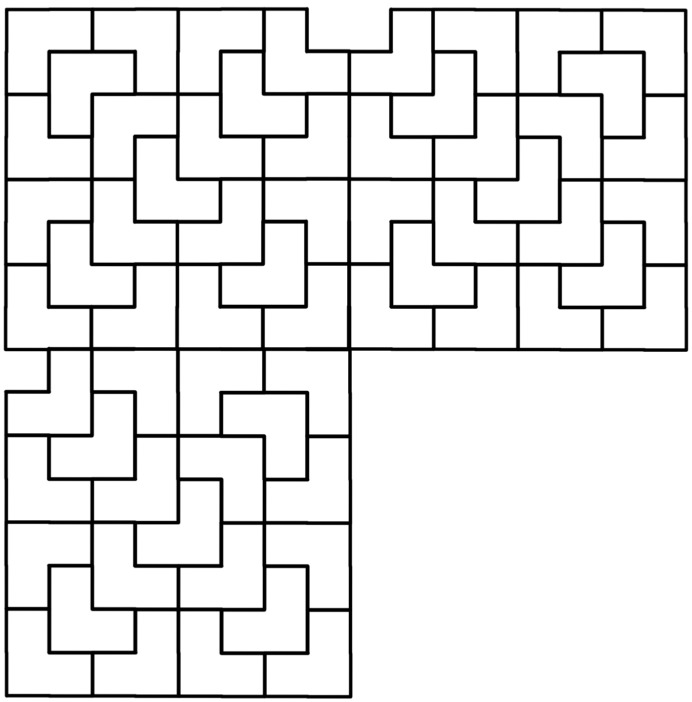
Λ is uniformly spread if it is BD to $\alpha \mathbb{Z}^d$ for some $\alpha > 0$.

Not all Delone sets are uniformly spread

$2\mathbb{Z}$ $\mathbb{Z}$

A Basic Result

Define a Delone set $\Lambda = \Lambda_T$ by picking one point in every tile in a tiling T of $\mathbb{R}^d$ that consists of copies of a single tile.

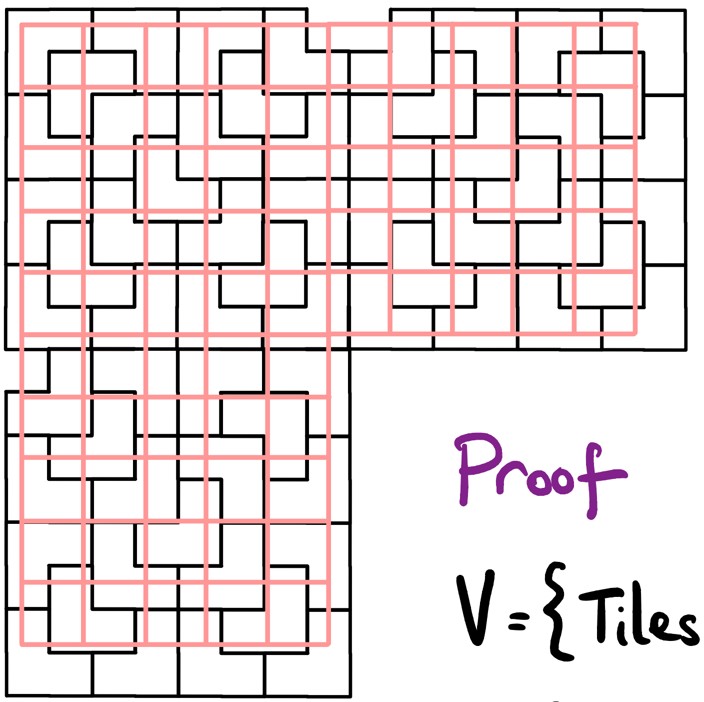


Folklore

Λ is uniformly spread

A Basic Result

Define a Delone set $\Lambda = \Lambda_T$ by picking one point in every tile in a tiling $\mathcal{T}$ of $\mathbb{R}^d$ that consists of copies of a single tile.



Folklore

Λ is uniformly spread

Proof define a bipartite graph G_Λ:

$V = \{\text{Tiles } T \text{ in } \mathcal{T}\} \dot\cup \{\text{Lattice cubes } C \text{ of volume vol}(T)\}$

$E = \{\{T, C\} \mid T \cap C \neq \emptyset\}$

If G_Λ contains a perfect matching

$\Rightarrow \Lambda$ is uniformly spread

Hall's Marriage Theorem (∞ version due to Rado)

A bipartite graph $G = (V_1 \dot\cup V_2, E)$ contains a perfect matching

$\iff$ $\forall$ finite subsets $F_1 \subset V_1$, $F_2 \subset V_2$:

$$\# F_1 \leq \# N_G(F_1) \quad \text{and} \quad \# F_2 \leq \# N_G(F_2)$$

Hall's Marriage Theorem (∞ version due to Rado)

A bipartite graph $G = (V_1 \dot\cup V_2, E)$ contains a perfect matching

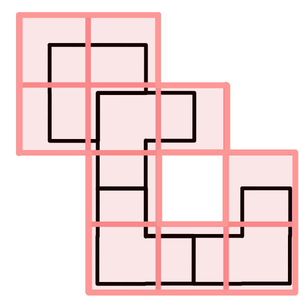
$\iff$ $\forall$ finite subsets $F_1 \subset V_1$, $F_2 \subset V_2$:

$$\# F_1 \leq \# N_G(F_1) \quad \text{and} \quad \# F_2 \leq \# N_G(F_2)$$

Pick a finite set F_1 of tiles in $\mathcal{T}$. Volumes of tiles and cubes are the same $\Rightarrow$ can't cover k of one with less than k of the other:

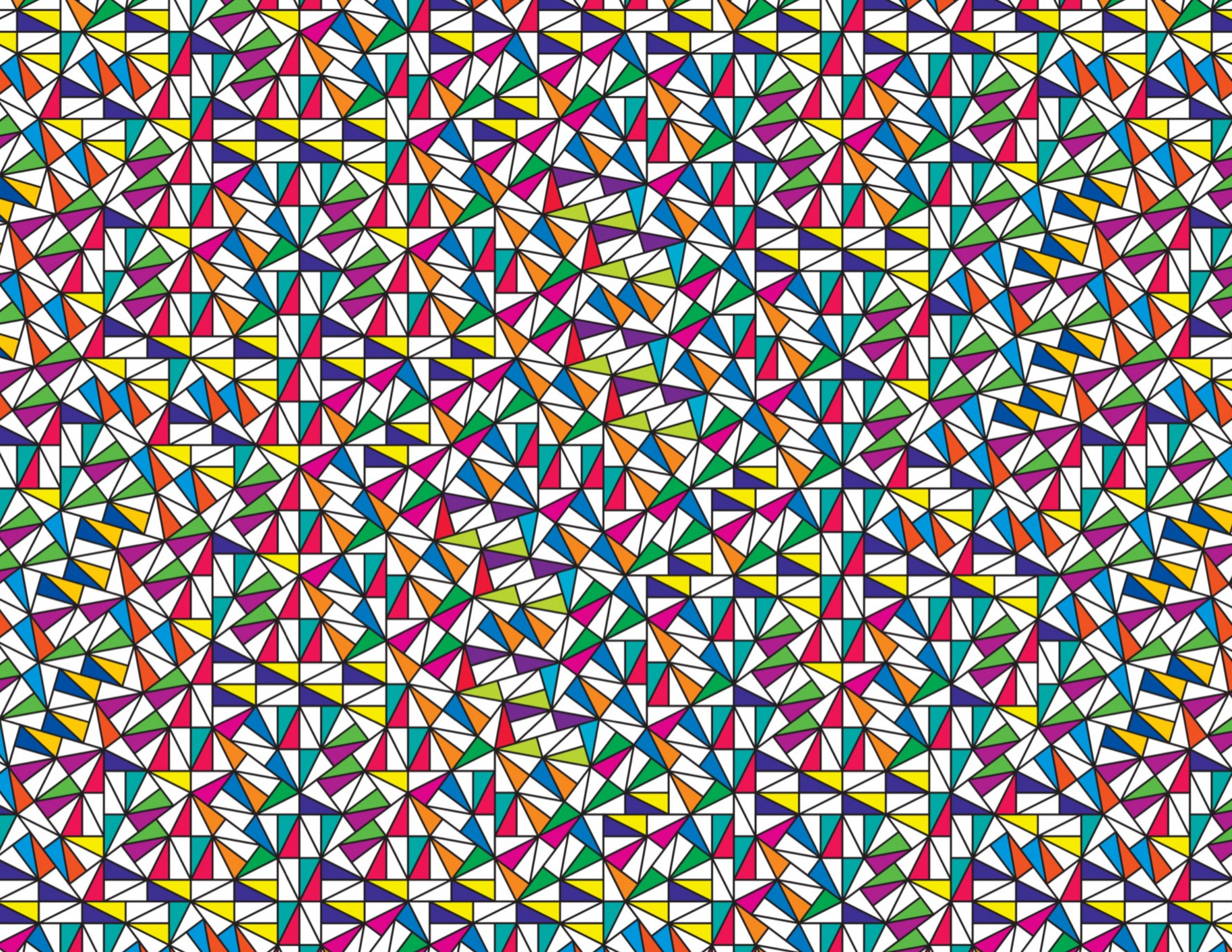
$$\# F_1 \leq \# \{\text{lattice cubes required to cover } F_1\}$$

$$= \# N_{G_\Lambda}(F_1) \qquad \left(\text{Similarly } \# F_2 \leq \# N_{G_\Lambda}(F_2)\right)$$



$\Rightarrow G_\Lambda$ contains a perfect matching and Λ is uniformly spread

Corollary lattices & periodic sets are unif. spread (Duneau, Oguey '90)
without Hall



Laczkovich Criterion

Using Hall's Theorem for BD is originally due to Laczkovich:

$\Lambda \overset{BD}{\sim} \Gamma \iff \exists\ m \in \mathbb{N}$ for which G_m contains a perfect matching, where

$$G_m = (V_1 \dot\cup V_2, E_m) = (\Lambda \dot\cup \Gamma, \{\{x,y\} \mid x \in \Lambda, y \in \Gamma, \|x-y\| \leq m\})$$

Laczkovich '92 For a Delone set $\Lambda \subset \mathbb{R}^d$ the following are equivalent:

- Λ is uniformly spread
- There exist $\alpha, C > 0$ so that $\forall\ A \in Q_d = \{$finite unions of lattice cubes$\}$

discrepancy $$|\#(A \cap \Lambda) - \alpha \cdot \mathrm{vol}(A)| \leq C \cdot \mathrm{vol}_{d-1}(\partial A)$$

Laczkovich Criterion

Using Hall's Theorem for BD is originally due to Laczkovich:

$\Lambda \overset{BD}{\sim} \Gamma \iff \exists\, m \in \mathbb{N}$ for which G_m contains a perfect matching, where

$$G_m = (V_1 \,\dot{\cup}\, V_2, E_m) = (\Lambda \,\dot{\cup}\, \Gamma, \{\{x,y\} \mid x \in \Lambda, y \in \Gamma, \|x-y\| \leq m\})$$

Theorem (FSS '21 and SS2 '21) The following are equivalent:

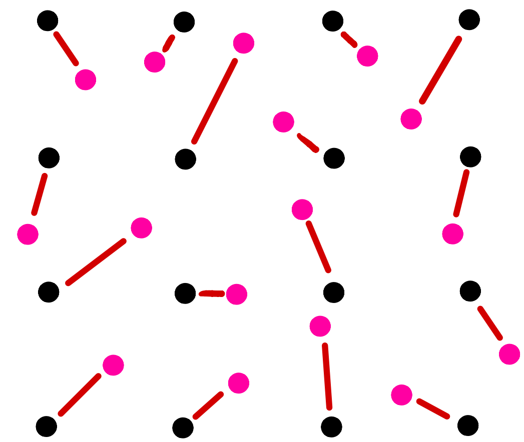
- Λ and Γ are not BD equivalent
- There exists a sequence of sets $A_m \in Q_d$ so that

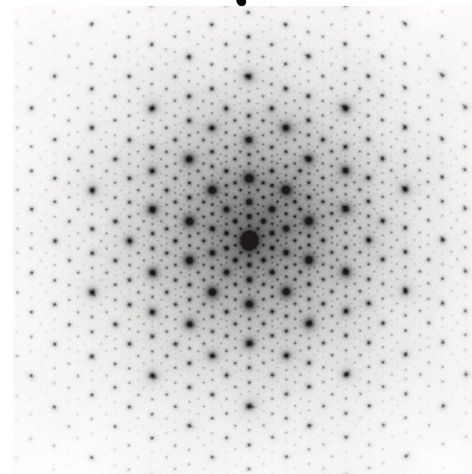
$$\frac{|\#(A_m \cap \Lambda) - \#(A_m \cap \Gamma)|}{\mathrm{vol}_{d-1}(\partial A_m)} \xrightarrow{m \to \infty} \infty$$

Our motivation is the study of constructions in aperiodic order.

Aperiodic Order

We are intersted in aspects of order and disorder in aperiodic sets



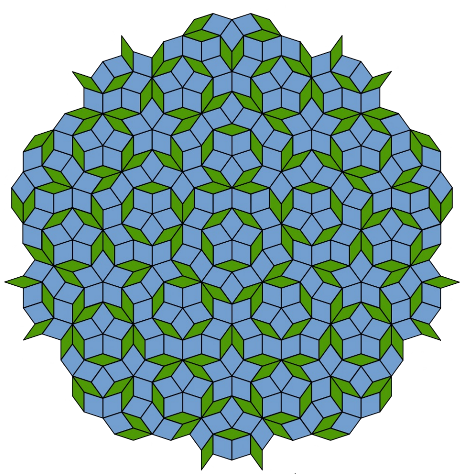


repetitivity

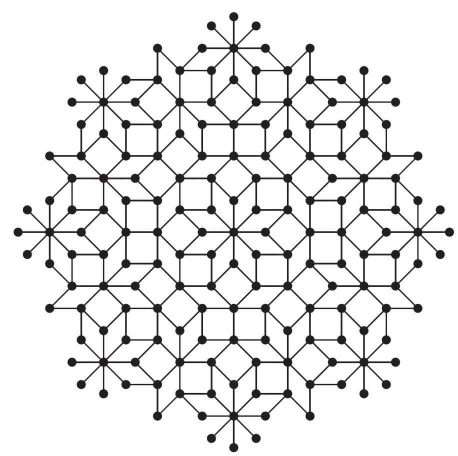
equivalence to lattices

sharp diffraction

and in constructions with interesting combinations of the above



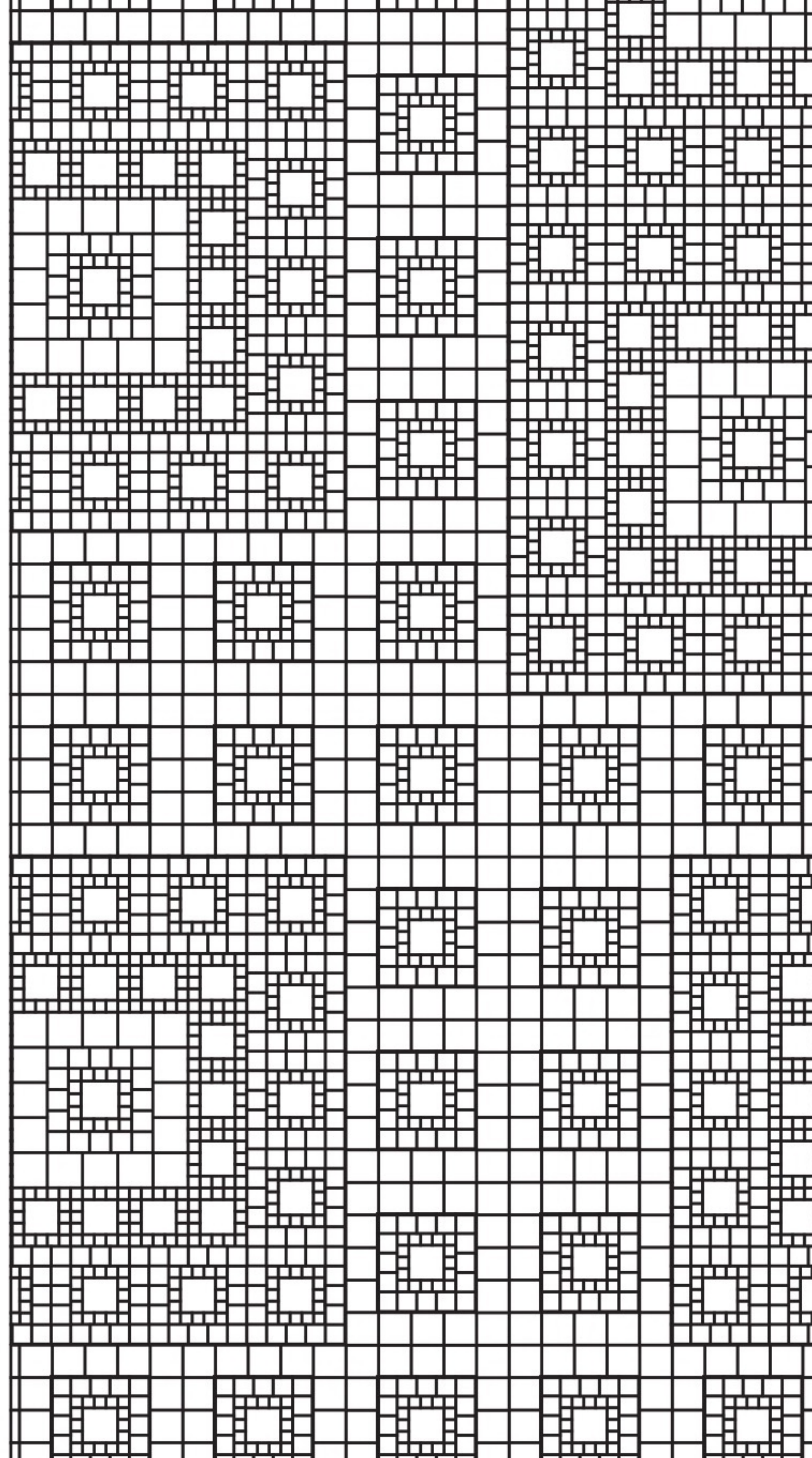
From Wikipedia



From Baake and Grimm's Aperiodic Order Vol 1

Plan of Talk

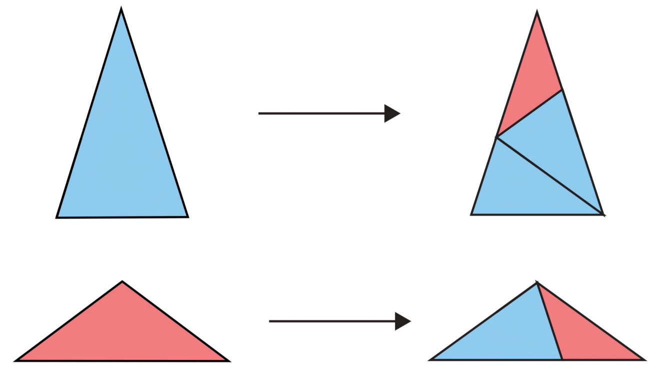
- Bounded displacement equivalence of point sets
- Substitution tilings
- Multiscale substitution tilings



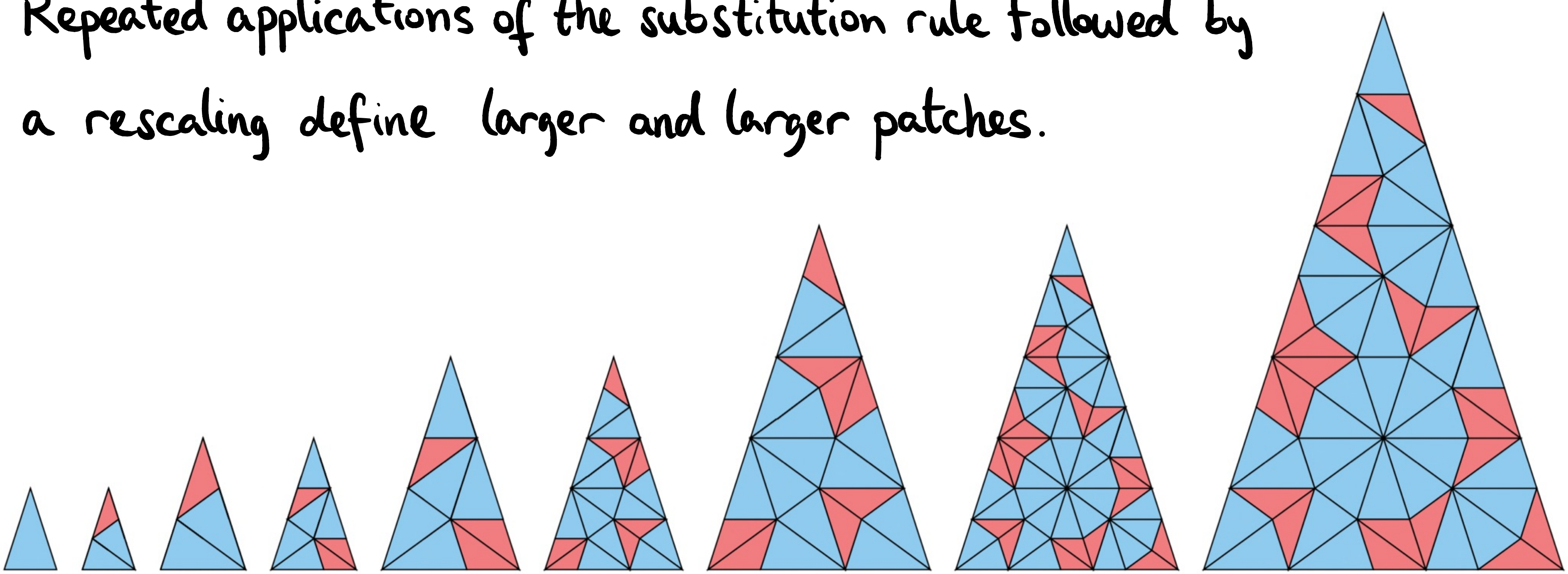
Substitution Tilings

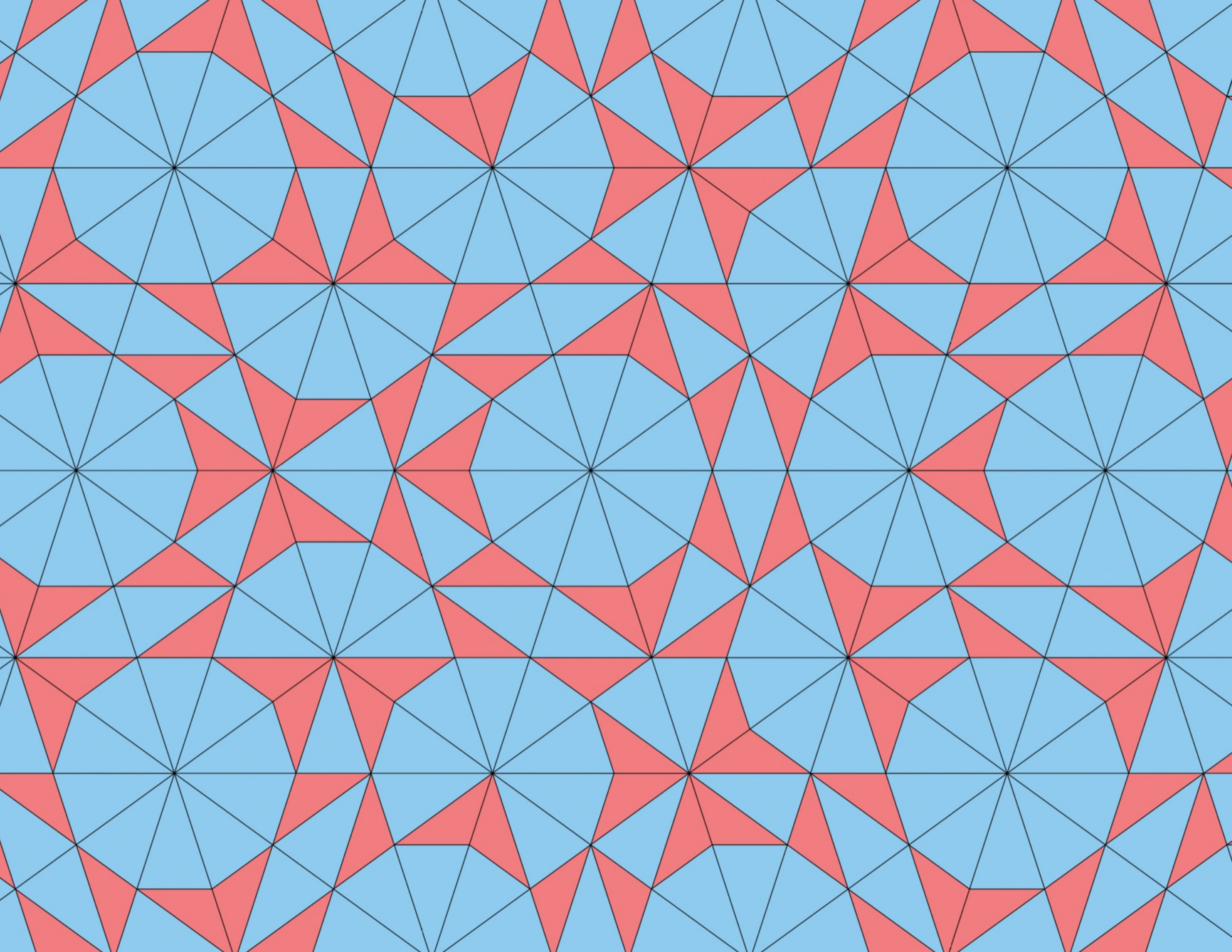
A **tiling** is a collection of tiles with disjoint interiors that covers $\mathbb{R}^d$.

A **substitution rule** on a set of **prototiles** is a tessellation of each prototile by rescaled prototiles, with a **fixed scale** $\in (0,1)$



Repeated applications of the substitution rule followed by a rescaling define larger and larger patches.





BD Equivalence For Substitution Tilings

Are Delone sets that correspond to Penrose tilings uniformly spread?

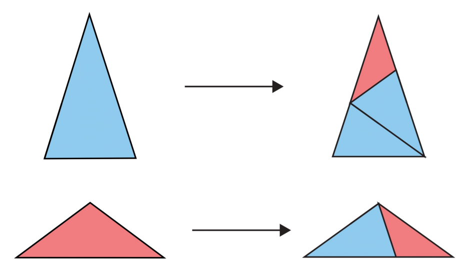
What can we expect for other substitution tilings?

Laczkovich: its a question of discrepancy $\left(|\#(A \cap \Lambda) - \alpha \cdot vol(A)| \overset{?}{\leq} C \cdot vol_{d-1}(\partial A) \right)$

$\Rightarrow$ A question of counting tiles.

A substitution rule defines

a substitution matrix S.



$\Rightarrow S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

\# tiles in patches = entries of powers of S

Perron-Frobenius Theorem: main term governed by leading eigenvalue

$\underbrace{\text{error term}}_{\text{discrepancy}}$ by smaller eigenvalues

BD Equivalence For Substitution Tilings

Theorem (Solomon '14) Let $j \geq 2$ be the minimal index with $V_{\lambda_j} \not\subseteq \mathbb{1}^{\perp}$, where $\lambda_1 > |\lambda_2| \geq \ldots \geq |\lambda_n|$ are the eigenvalues of S. Then

- $|\lambda_j| < \lambda_1^{\frac{d-1}{d}} \Rightarrow$ Corresponding Delone sets are uniformly spread.
- $|\lambda_j| > \lambda_1^{\frac{d-1}{d}} \Rightarrow$ Corresponding Delone sets are not uniformly spread.

(Penrose $S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $\frac{1}{\varphi^2} < \varphi = [\varphi^2]^{\frac{2-1}{2}} \Rightarrow$ uniformly spread)

BD Equivalence For Substitution Tilings

Theorem (Solomon '14) Let $j \geq 2$ be the minimal index with $V_{\lambda_j} \not\leq \mathbb{1}^{\perp}$, where $\lambda_1 > |\lambda_2| \geq \ldots \geq |\lambda_n|$ are the eigenvalues of S. Then

- $|\lambda_j| < \lambda_1^{\frac{d-1}{d}} \Rightarrow$ Corresponding Delone sets are uniformly spread.
- $|\lambda_j| > \lambda_1^{\frac{d-1}{d}} \Rightarrow$ Corresponding Delone sets are not uniformly spread.

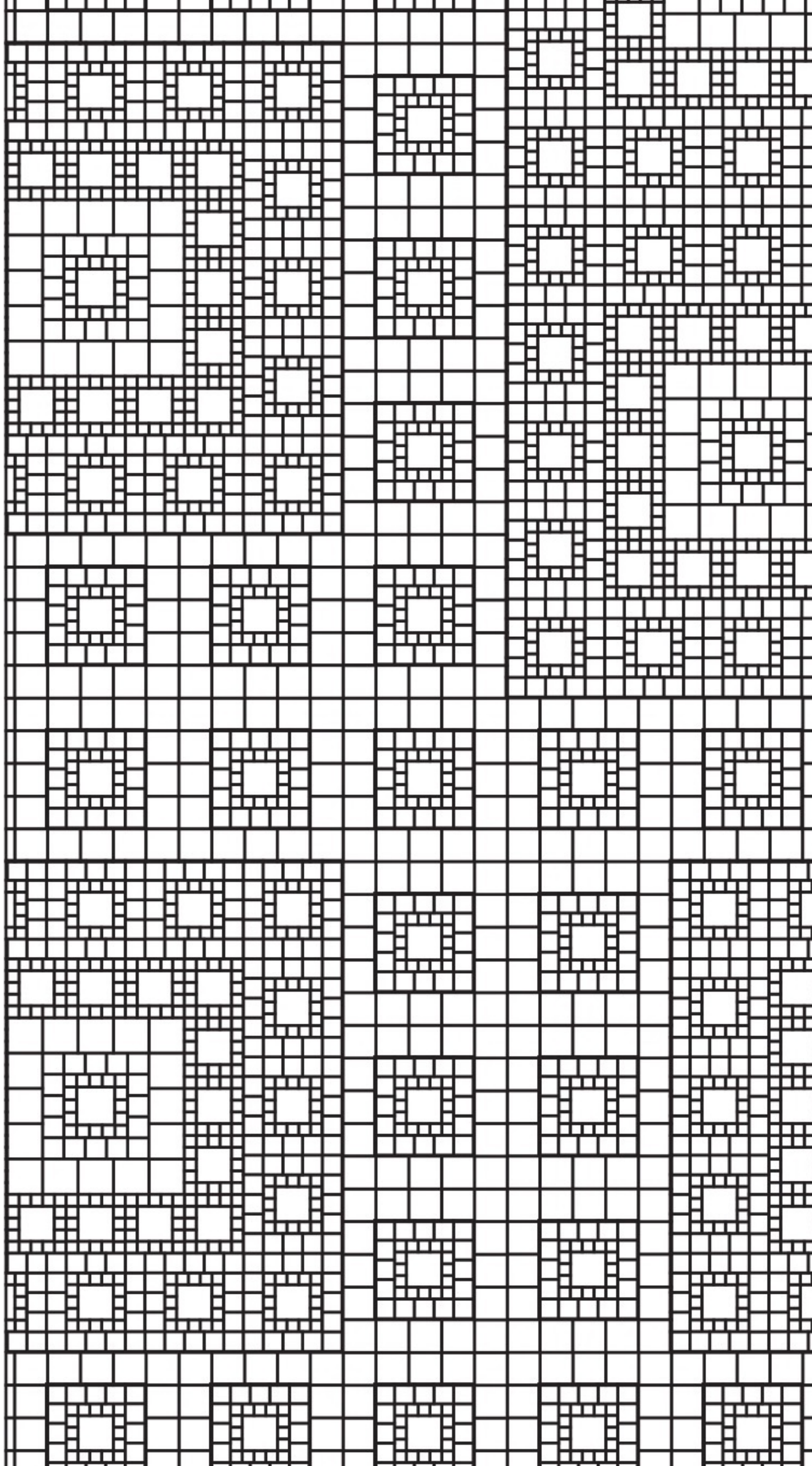
(Penrose $S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ $\frac{1}{\varphi^2} < \varphi = [\varphi^2]^{\frac{2-1}{2}} \Rightarrow$ uniformly spread)

Theorem (FSS '21) $\exists$ matrices with $|\lambda_j| = \lambda_1^{\frac{d-1}{d}}$ that are associated both to periodic and to non-uniformly spread tilings.

Theorem (SS2 '21) In the non-uniformly spread case the substitution rule generates tilings in continuously many BD classes.

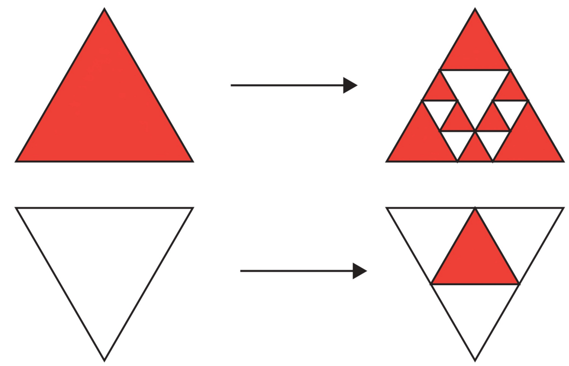
Plan of Talk

- Bounded displacement equivalence of point sets
- Substitution tilings
- Multiscale substitution tilings

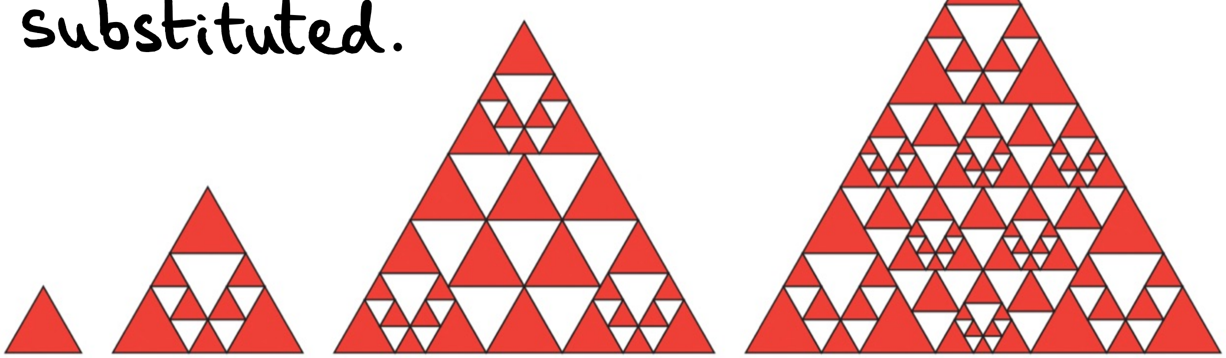


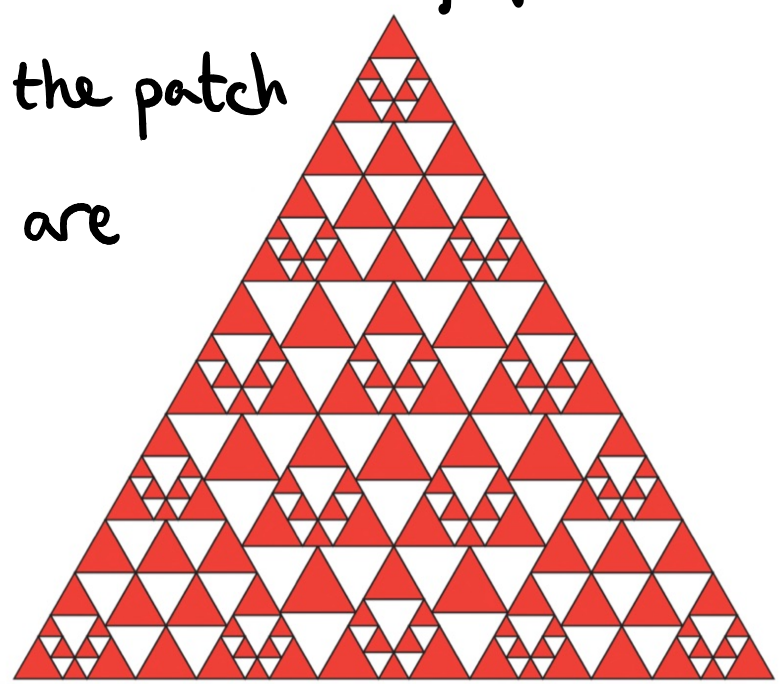
Incommensurable Multiscale Substitution Tilings (SS1 '21)

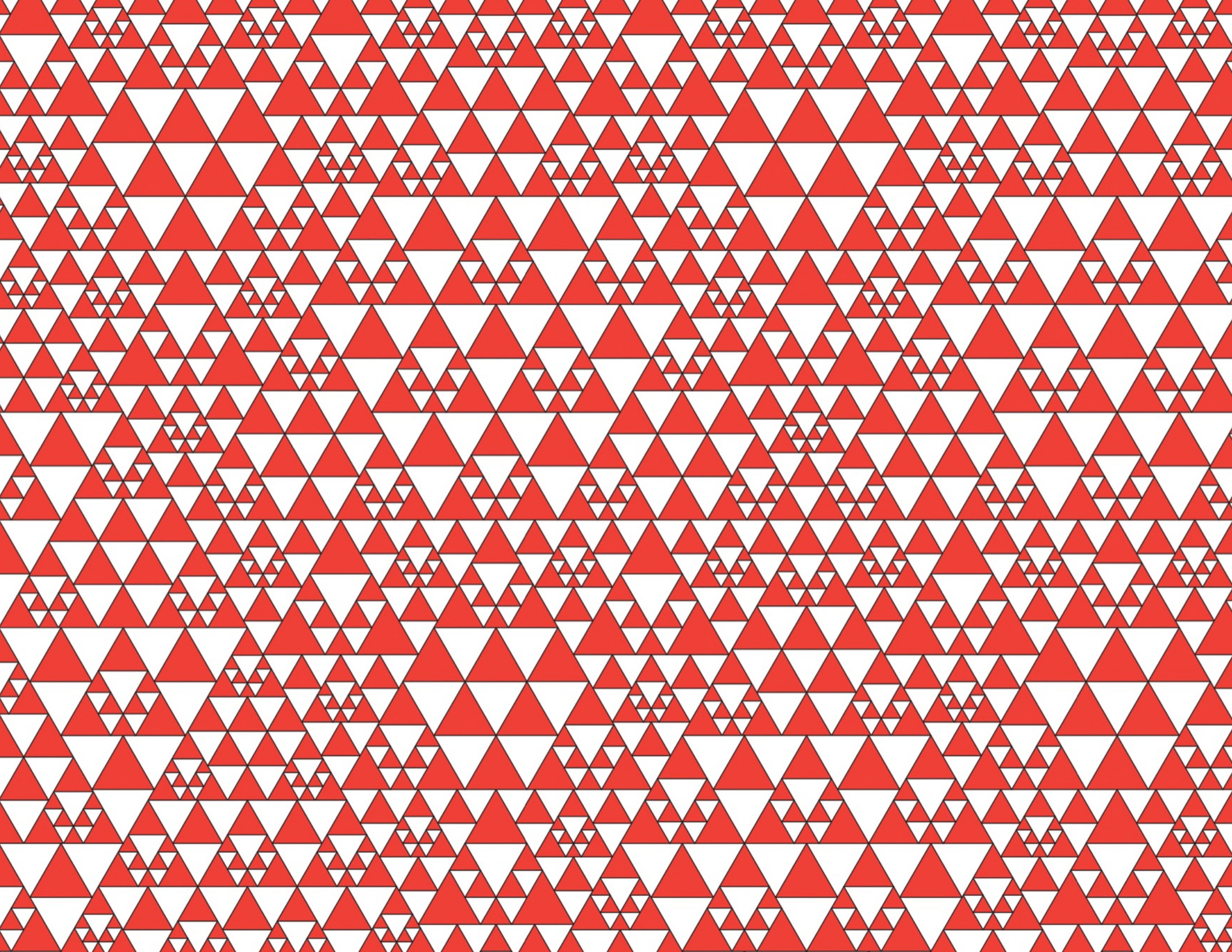
A multiscale substitution scheme σ in $\mathbb{R}^d$ consists of a substitution rule on unit volume prototiles $T_1, \ldots, T_n$, where various different scales appear and satisfy a simple incommensurabily condition.



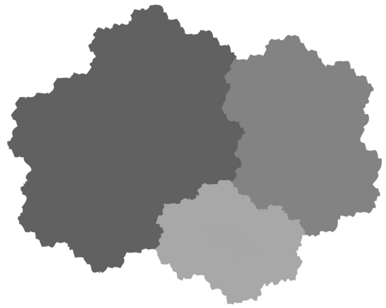
A time-dependant substitution semiflow F_t defines a family of patches: At time $t=0$ $F_t(T) = T$, and as t increases the patch is inflated by e^t and tiles of volume > 1 are substituted.

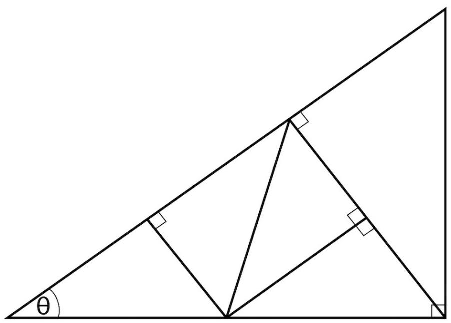






Some Predecessors

- Rauzy's fractal '81  multiple (but commensurable) scales



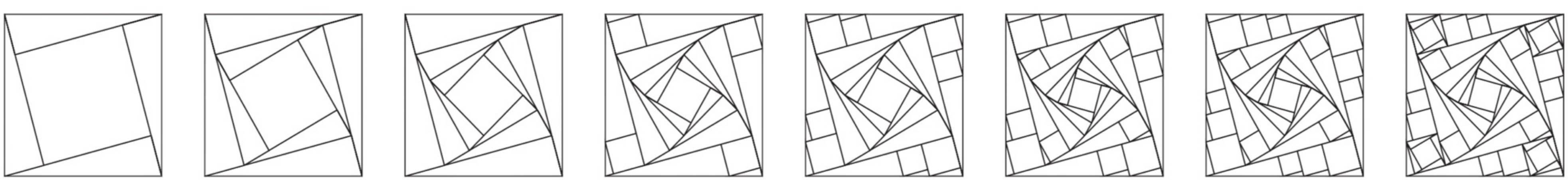
- Conway and Radin's pinwheel tiling '94

$\theta = \arctan \frac{1}{2} \Rightarrow$ same triangle incommensurable directions

- Sadun's generalized pinwheel tilings '98

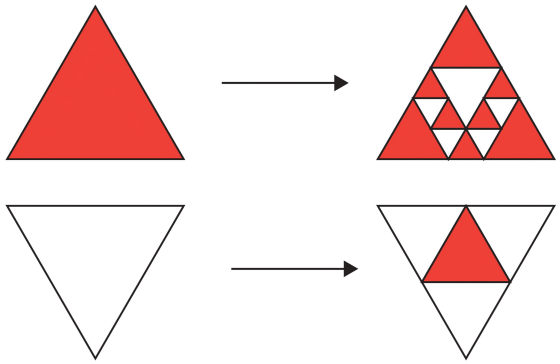
- α-Kakutani sequences in $[0,1]$ '76  always split longest interval

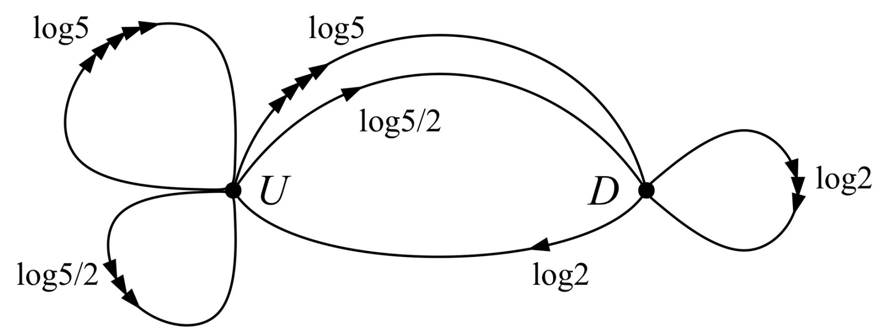
- S '20: multiscale substitution Kakutani sequences of partitions



The Associated Graph G_σ

A directed weighted graph is defined according to σ





Vertices model the prototiles

Edges model the tiles appearing in the substitution rule with

Lengths $= \log(1/\text{scale})$

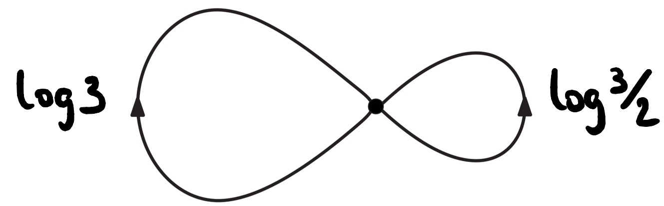
σ is incommensurable if G_σ contains two closed paths of lengths $\frac{a}{b} \notin \mathbb{Q}$.

Incommensurable multiscale substitution schemes generate a new distinct class of tilings of $\mathbb{R}^d$.

Counting in Multiscale Substitution Tilings

$$\{\text{Tiles in } F_t(T_i)\} \longleftrightarrow \left\{\begin{array}{l}\text{Directed walks of length } t \\ \text{in } G_\sigma \text{ originating at vertex } i\end{array}\right\}$$

Example the $\frac{1}{3}$- Kakutani scheme in $\mathbb{R}$:



the patches $F_0(I), F_{\log\frac{3}{2}}(I), F_{2\log\frac{3}{2}}(I)$ are given with their respective walks

$\Rightarrow$ Discrepancy requires counting walks on incommensurable graphs and evaluating error terms

Counting in Multiscale Substitution Tilings

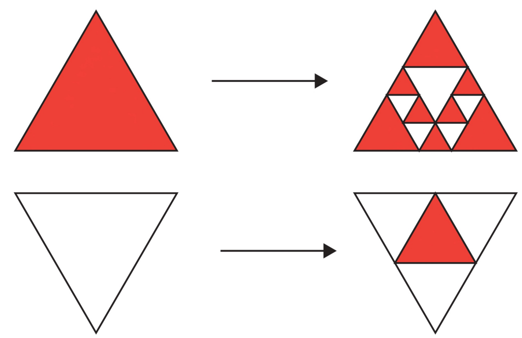
Theorem (S ≥'21, relying on Kiro, Smilansky×2 '20)

$$\#\{\text{tiles in } F_t(T)\} = \frac{v^T(S_\sigma - V_\sigma)\mathbb{1}}{v^T H_\sigma \mathbb{1}} \cdot \underbrace{e^{dt}}_{\mathrm{vol}(F_t(T))} + \text{ERROR TERM}, \quad t \to \infty$$

combinatorics matrix $\quad (S_\sigma)_{ij} = \sum_{\substack{T \text{ of type } j \\ \text{in } T_i}} 1$

\# reds in white

$$S_\sigma = \begin{pmatrix} 8 & 5 \\ 1 & 3 \end{pmatrix}$$



volume matrix $\quad (V_\sigma)_{ij} = \sum_{\substack{T \text{ of type } j \\ \text{in } T_i}} \mathrm{vol}(T)$

total red area in white

$$V_\sigma = \begin{pmatrix} \frac{17}{25} & \frac{8}{25} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

entropy matrix $\quad (H_\sigma)_{ij} = \sum_{\substack{T \text{ of type } j \\ \text{in } T_i}} -\mathrm{vol}(T) \cdot \log \mathrm{vol}(T)$

contribution of reds to entropy of white

$$H_\sigma = \begin{pmatrix} -\frac{12}{25}\log\frac{4}{25} - \frac{5}{25}\log\frac{1}{25} & -\frac{4}{25}\log\frac{4}{25} - \frac{4}{25}\log\frac{1}{25} \\ -\frac{1}{4}\log\frac{1}{4} & -\frac{3}{4}\log\frac{1}{4} \end{pmatrix}$$

and v^T = left Perron-Frobenius eigenvector of V_σ

Counting in Multiscale Substitution Tilings

Theorem (SS1 '21) $\exists k \in \mathbb{N}$ $\forall t_0 > 0$ $\exists t \geq t_0$: ERROR TERM $\geq C \dfrac{\mathrm{vol}(F_t(T))}{t^k}$

$$\Bigg|\underbrace{\#\{\text{tiles in } F_t(T)\}}_{\#(\Lambda \cap A)} - \underbrace{\frac{v^T(S_\sigma - V_\sigma)\mathbb{1}}{v^T H_\sigma \mathbb{1}} \mathrm{vol}(F_t(T))}_{\alpha \cdot \mathrm{vol}(A)}\Bigg| \geq \underbrace{C\frac{e^{dt}}{t^k}}_{\geq C \mathrm{vol}_{d-1}(A)}$$

$\Rightarrow$ Incommensurable tilings are never uniformly spread.

Idea of proof by the combinatorial structure of G_σ $\exists \leq \sim t^k$ distinct volumes of tiles in $F_t(T)$, $k = \#$ edges in G_σ. $\exists \sim \mathrm{vol}(F_t(T))$ tiles in $F_t(T)$ and by the pigeonhole principle $\exists$ a volume shared by $\geq \sim \dfrac{\mathrm{vol}(F_t(T))}{t^k}$ tiles, which will be substituted simultaneously.

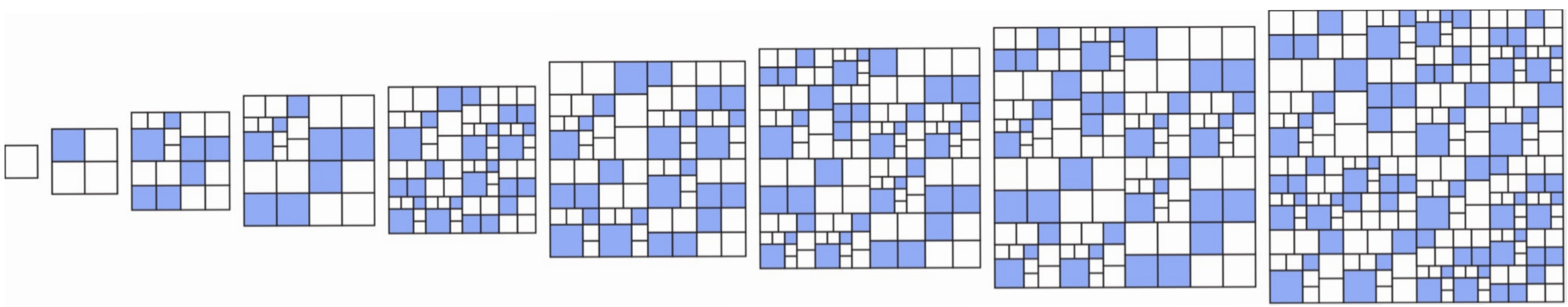
Theorem (SS2 '21) Every incommensurable multiscale substitution rule generates tilings in continuously many BD classes.

Counting in Multiscale Substitution Tilings

Theorem (S≥'21, relying on Kiro, Smilansky×2 '20)

Similar asymptotic formulas for:

- $\#\{$tiles of type r and $vol \in [a,b]$ in $F_t(T)\}$
- volume$\left(\bigcup \{\text{tiles of type } r \text{ and } vol \in [a,b] \text{ in } F_t(T)\}\right)$
- Expected values for random partitions



Counting in Multiscale Substitution Tilings

Theorem (S≥'21, relying on Kiro, Smilansky×2 '20) similar formulas for

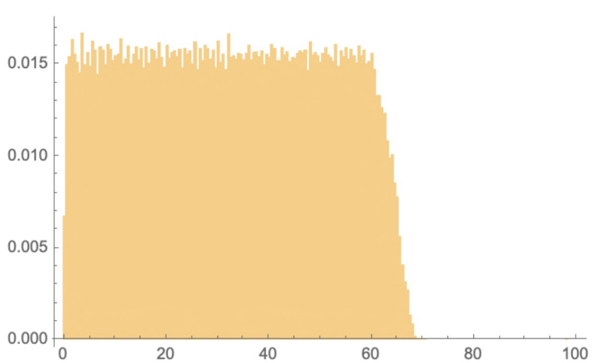
- **Gap distribution** Λ = Delone set of tile boundaries in a 1-dim tiling

$$\frac{\#\{\text{Neighbors in } \Lambda\cap[-N,N] \text{ of distance} \in [a,b]\}}{\#\{\Lambda\cap[-N,N]\}} \longrightarrow \int_a^b \frac{v^T C_\sigma(x)\,\mathbb{1}}{v^T H_\sigma \mathbb{1}}\,dx$$

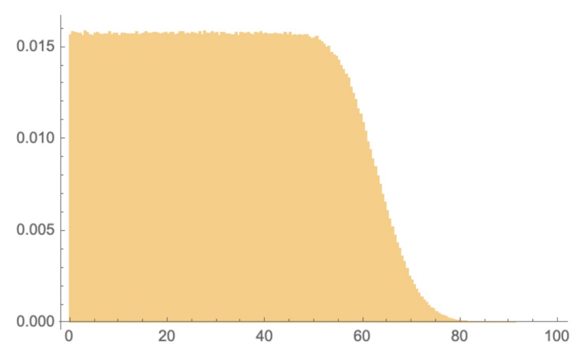
where $$(C_\sigma(x))_{ij} = \sum_{\substack{T \text{ of type } j\\ \text{in } T_i}} \begin{cases} \dfrac{\mathrm{vol}\,T}{x^2}, & \mathrm{vol}\,T < x \le 1 \\ 0, & \text{otherwise}\end{cases}$$

- **Numerics** for **pair correlations** are consistent with **Poisson process**

```
list = {0, 3^10}; i = 1;
Do[While[list[[i + 1]] - list[[i]] > 1,
  list = Insert[list, list[[i]] + (list[[i + 1]] - list[[i]]) / 3, i + 1]], {i, 91 005}];
gaps = Flatten[Table[N[Differences[list, 1, j]], {j, 1, 100}]];
Histogram[gaps, {0, 100, 0.5}, "PDF"]
```

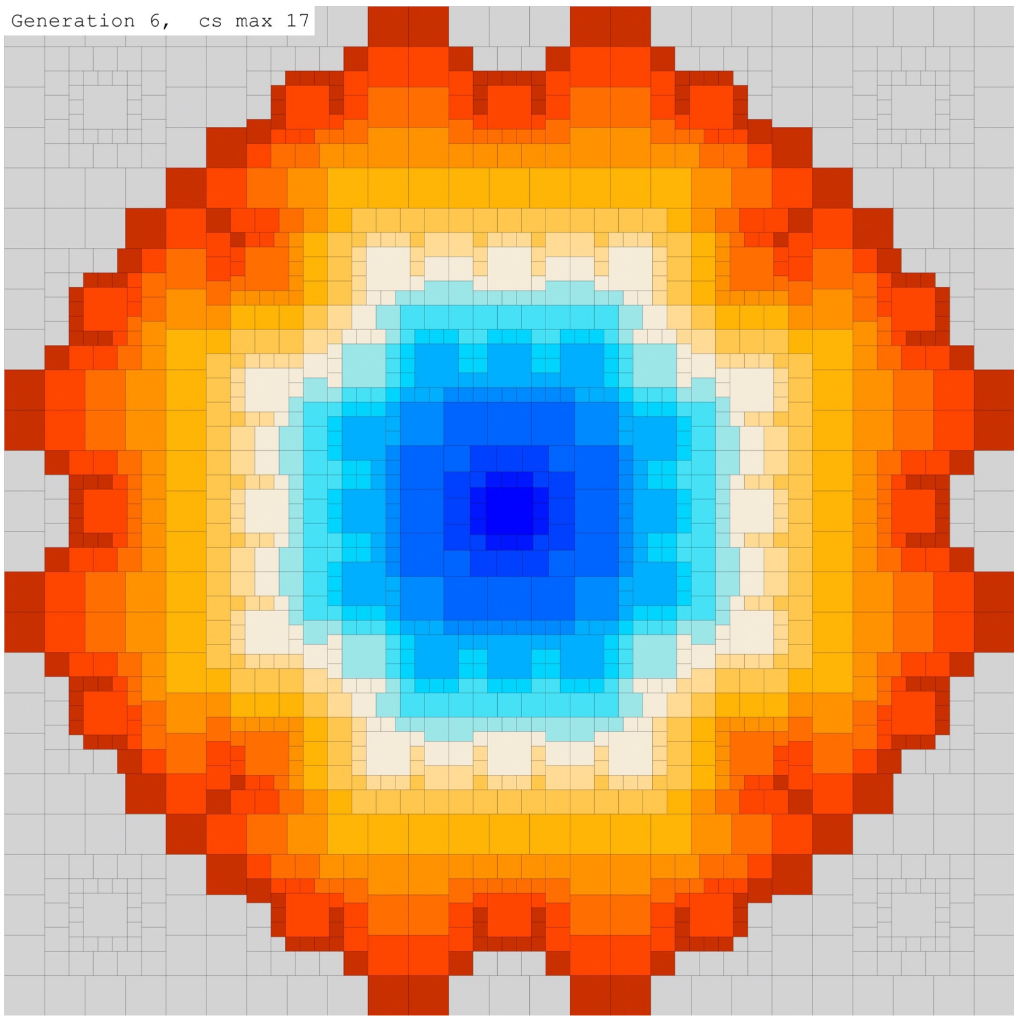


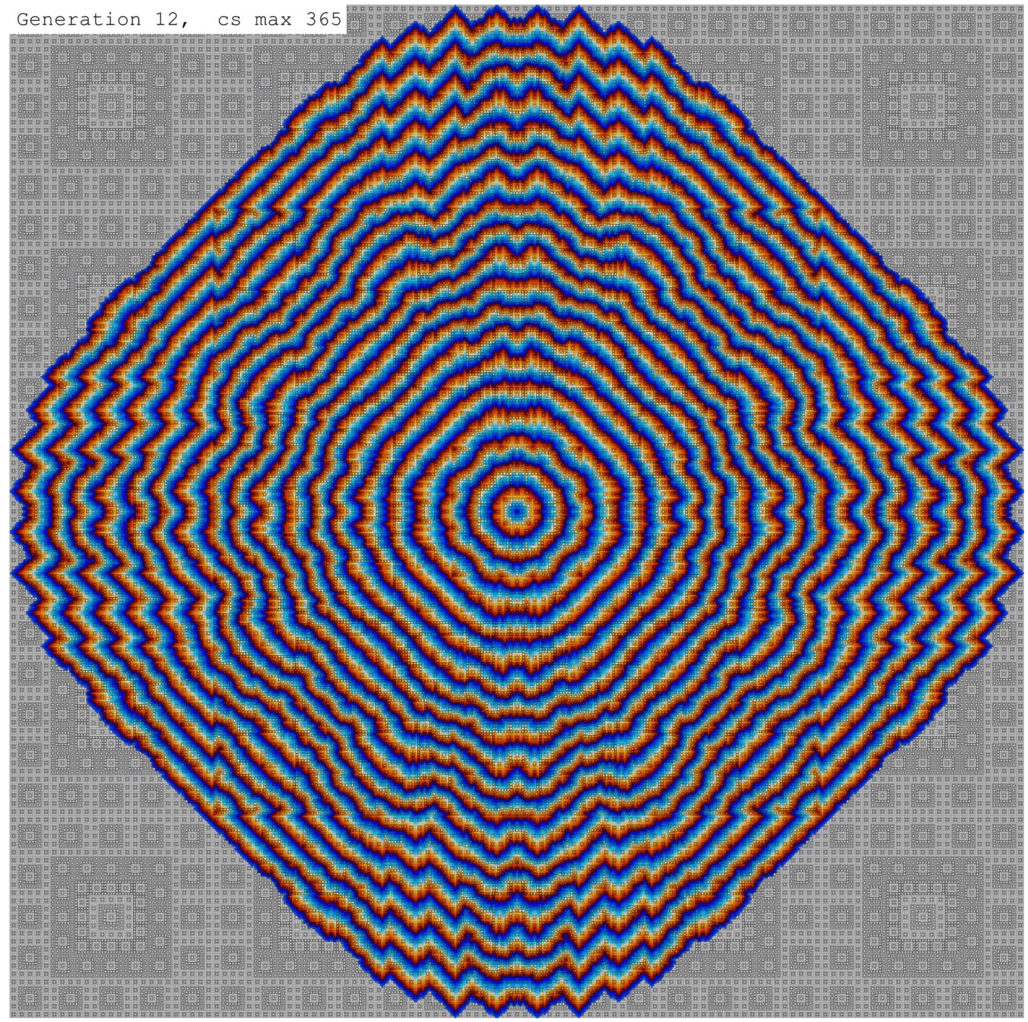

```
averagegap = 1 / (-(1 / 3) * Log[1 / 3] - (2 / 3) * Log[2 / 3]);
list = Accumulate[RandomVariate[ExponentialDistribution[averagegap], 90 000]];
gaps = Flatten[Table[N[Differences[list, 1, j]], {j, 1, 100}]];
Histogram[gaps, {0, 100, 0.5}, "PDF"]
```



Coordination Sequences

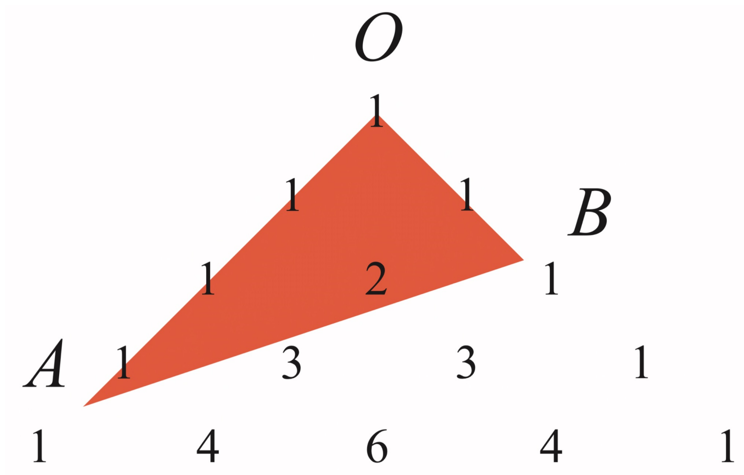
$a(n) :=$ # tiles of distance n from center $= 1, 12, 16, 16, 40, \ldots$



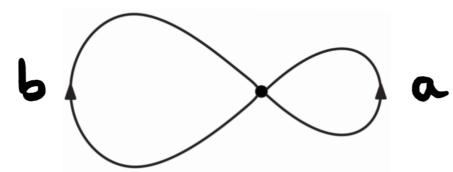


Courtesy of Lars Blomberg (OEIS sequence A328074)

Pascal's Triangle



Sum of binomial coefficients inside triangle of sides $OA = \frac{x}{a}$ $OB = \frac{x}{b}$ = # paths of length $\leq x$ in



Theorem (Kiro, Smilansky×2 '20) If $a \notin \mathbb{Q} b$ then this is

$$\frac{1}{\lambda a e^{-\lambda a} + \lambda b e^{-\lambda b}} e^{\lambda x} + o(e^{\lambda x}), \quad x \to \infty$$

where $\lambda > 0$ is the unique real zero of $f(s) = 1 - e^{-as} - e^{-bs}$.

Thank You!